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A Low-Achiever's Learning Process in Mathematics: Shirley's Fraction Learning

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ABSTRACT

Research in mathematics education offers a considerable body of evidence that both high and low-achievers can benefit from learning mathematics in meaningful contexts. This case study offers an in-depth analysis of the learning process of a low-achieving student in the context of Realistic Mathematics Education (RME). The focus is on the use of productive and counter productive strategies in learning fractions. We found support for our idea that low-achievers do benefit from RME, but experience difficulties in the formalization process with regard to fractions. We seize upon the observed difficulties by discussing the implications of uniform standards in mathematics education.

INTRODUCTION

In this article we describe a case study of a low performing student in mathematics who learned fractions in a newly designed program. In this, fraction teaching found a place in a setting where meaning was negotiated and where the number line and the bar were central models to support the students in developing fraction strategies. We reported elsewhere on the development of the program and its impact on the learning processes and outcomes of students in grade 6 (9-10 years) (Keijzer, 1994; Keijzer & Lek, 1995; Keijzer & Buys, 1996; Keijzer, 1997; Keijzer, 1999; Keijzer & Terwel, 2000; Keijzer & Terwel, 2001). We found that students in this experimental program outperformed students in a control group, where the students mainly worked individually and where the circle was the major fraction model (Keijzer & Terwel, 2000; Keijzer & Terwel, 2003).

The student we write about in the present study, Shirley, belongs to the 25 percent weakest in mathematics in her age group, as judged within a national context (Janssen, Kraemer & Noteboom, 1995). Janssen, Kraemer & Noteboom consider students in this group to be low-achievers that need special teacher's attention. In the present case study we observe low-achiever Shirley having major difficulties in learning fractions and answer the question why fractions are that difficult for Shirley. Moreover, regarding the limitations of case-study research, we consider to what extent findings concerning Shirley can be generalized for low-achievers like her.

This case study is part of a larger research project in which we analyzed the development of grade 6 students (9-10 years) learning fractions in two different programs (Keijzer & Terwel, 2000; Keijzer & Terwel, 2003; Keijzer, 2003). All students in this project were observed meticulously during their fraction lessons. In these observations and in our analyses of test results, we noticed a so-called "Matthew-effect"; the strong students grew stronger, while the poor performers in mathematics stayed behind (Kerckhoff & Glennie, 1999). For this reason we tried to uncover the processes by which low-achievers in mathematics end up with disadvantaged outcomes.

Theoretical Background

Low-Achievers in Mathematics Education

Many researchers in mathematics education focus on learning processes of low-achievers in mathematics. There is some evidence from the literature that the process of learning arithmetic for these students is different from that of students who perform normally (Van Lieshout, 1997; Milo, 2003). Others (e.g. Kraemer & Janssen, 2000; Kraemer, 2000) argue that the learning of arithmetic of low-achievers in mathematics is different from their more advanced classmates because these low achieving students lack a repertoire of context-bound mathematical relations and therefore experience difficulties when there is a need for considering numbers as formal objects, as they miss the reference to the contexts which embedded these more formal objects and relations. Hoek, Terwel and Van der Eeden (1997) found similar results in their research of interaction processes in cooperative groups in secondary mathematics. Moreover, they found an additional mechanism that disadvantaged low-achievers as help seekers: "Low-achievers are not always able to ask for the right help, because it is difficult for them to explain what they do not understand." (Hoek, Terwel & Van der Eeden, 1997, p. 366).

According to Sweller's "cognitive load" theory (1994, 1999), it is conceivable that low-achievers may also have problems with memory capacity, especially in solving complex problems in real-life situations. These problem situations are essentially "ill structured" and require flexible problem-solving strategies. Therefore the learning of a new mathematical topic, for example fractions, as it necessitates the use of new problem-solving strategies and the many factors represented in an ill-structured problem situation exceed the limits of their memory capacity. These memory problems are even increased by inefficient management of their memory capacity, especially in the context of a less efficient problem representation and a more complex road to the solution (Pollock, Chandler & Sweller, 2002; Krutetskii, 1976; Reigeluth, 1999; Hoek, Van den Eeden & Terwel, 1999).

Kraemer and Janssen (2000) indicate, however, that low-achievers can achieve number relations if these are given sufficiently lengthy explanations in meaningful, recognizable and identifiable contexts.

Fractions and Realistic Mathematics Education

The fractions curriculum considered here was developed as an extension of the "Fractiongazette" [De Breukenbode (Bokhove, et al., 1996)]. Both the Fractiongazette and its extension were developed within the Dutch context of realistic mathematics education (RME). This implies that recognizable and meaningful contexts are used to help students build upon their informal knowledge. Moreover, these contexts lead to modeling, schematizing and hence to the construction of formal relations between numbers and other mathematical objects (Treffers, 1987; Freudenthal, 1991; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996).

Starting from the RME paradigm, the constructed extension of the Fractiongazette is marked by whole class discussions in which meaningful fractions are negotiated and constructed. Thus the teaching may be characterized as interactive and reflective. The curriculum is directed towards the acquisition of number sense (Greeno, 1991; Mcintosh, Reys & Reys, 1992), that is, students learn to attach meaning to fractions in various kinds of situations, develop a good notion of the size of fractions, and learn to handle fractions in simple applications.

Recent research in the Dutch context of RME offers a considerable body of evidence that both high- and low-achievers are helped by acquiring mathematics through problem solving in meaningful contexts (Van den Heuvel-Panhuizen, 1996; Van Luit & Van de Rijt, 1997; Hoek, Terwel & Van der Eeden, 1997; Kraemer & Janssen, 2000; Kraemer, 2000). By learning fractions in meaningful situations in this RME-context, the fraction program discussed here might remove some of the difficulties mentioned by Hasemann (1981). By using meaningful contexts, the students develop "fraction language," meaning that they can connect fractions in written or oral form to a division, like "two thirds" means "you have to divide in three and take two." Furthermore, the contexts are chosen in such a way that the number line as a model for fractions comes into sight (cf. Moss & Case, 1999). As regards the fractions that belong in the same position on the number line, equivalent fractions are highlighted, laying a base for formal manipulations with fractions.

Although results of RME in this respect are remarkable, it has its limitations. For instance, one could argue that these contexts, which are meaningful to most students, might be meaningless to many low-achieving students. And, if low-achievers come to regard a context which is meant to be meaningful as just another confusing mathematical artifact, learning fractions could easily degenerate into a mechanical application of the rules of arithmetic (cf. Erlwanger, 1973).

An Experimental Curriculum

The curriculum considered here contains a four-stage teaching strategy in which number sense is developed by: (i) providing a language of fractions, (ii) developing the number line for fractions, (iii) comparing fractions, (iv) learning formal fractions. In the curriculum, different situational contexts and models are used. Two types of fraction-generating activities, dividing and measuring, lead to the bar and the number line as central fraction models. In the teaching of the curriculum students are offered the opportunity to present their approaches at several levels.

Problem Solving

Several researchers (e.g. Behr, et al., 1983) show that teaching fractions in the way described, with students constructing their own higher level fraction relationships, inevitably leads to problem solving, where problem solving may be characterized as consisting of all those heuristic approaches to mathematical solutions in which the problem solver has no direct algorithmic approach available. Verschaffel (1995) points to three types of knowledge needed for problem solving:

1. the flexible use of a rich and well organized, domain-specific knowledge base;
2. the ability to use heuristic methods;
3. metacognition.

Verschaffel emphasizes that these points represent difficulties especially for those students who are weak in mathematics. In combination with the argument advanced by Behr et al. that constructing higher level fraction relationships can be regarded as problem solving, Verschaffel explains why fractions are so difficult for low-achievers.

A similar argument is presented in Nelissen (1998a). Nelissen takes the formation of representations as the starting-point for his argument: "By reflecting on their own actions, children can construct representations on a higher level, requiring critical testing."[1] (Nelissen, 1998a, p. 175)

Nelissen specifies the problem solving process for both high- and low-achievers. In solving mathematical problems, low-achievers, once they have found a (usually standard) approach to a solution, in general hold on to it. By contrast, high-achievers in general dare to change their strategy and abandon a chosen solution where appropriate. In addition, Lemoyne and Tremblay (1986) characterize good problem solvers as students with rich and precise associations. They argue that problem-solving strategies largely depend on linguistic and heuristic strategies. This links up with Nelissen (1998b) who argues that both the learning of language and the learning of mathematics are characterized by the use of representations. He shows what makes learning mathematical language so difficult:

> In daily life ambiguities [in natural language] do not trouble us, because there are many situational cues. In mathematics classes, however, children are often unprepared when confronted with words such as: table, times, angle, magnitude, power, set, small number, operation, match, dividing, etc.[2] (Nelissen, 1998b, p.15/6)

Booth and Thomas (2000) add yet another argument to the difficulty of problem-solving tasks for low-achievers in mathematics. They state that it is easier for students to use visual representations of problems that are context-near, such as drawings, than more developed representations such as schemata and models. They argue that more developed (formal) representations of this kind might cause problems for low-achievers.

Turning Tide for Low-Achievers

In summary, the aforementioned researchers observed the following characteristics in low-achievers' cognitive functioning:

- low-achievers often undertake long and complex searches and lack the metacognitional strategies to escape from solutions that work elsewhere (Krutetskii, 1976; Verschaffel, 1995);
- low-achievers have problems with cognitive overload, especially in solving complex problems in real life situations (Krutetskii, 1976; Sweller, 1994; Sweller, 1999);
- low-achievers lack the flexible use of a rich and well organized, domain-specific knowledge base (Verschaffel, 1995; Lemoyne & Tremblay, 1998; Kraemer, 2000);
- low-achievers lack the ability to use heuristic methods (Verschaffel, 1995; Nelissen, 1998a);
- low-achievers have difficulties in understanding more developed representations like schemata and models (Nelissen, 1998a; Booth & Thomas, 2000); they also experience difficulties in developing mathematical language (Nelissen, 1998b);
- low-achieving students experience difficulties when there is a need for considering numbers as formal objects and there is no clear reference to the contexts that produced them (Kraemer & Janssen, 2000);
- as a consequence, low-achievers are not always able to ask for the right form of assistance because it is difficult for them to explain what it is they do not understand (Hoek, Terwel & Van der Eeden, 1997);
- finally, they face dropping out (Holt, 1964).

Although these researchers provide a considerable body of evidence that low-achievers experience major difficulties in problem-solving situations, many others find arguments in the nature of problem solving to make this the starting-point of their mathematics education. These researchers thus challenge the deficit approach of low-achievers by actively searching for means to value low-achievers' social and cognitive possibilities. Kraemer (2000) found that low-achievers in mathematics are best helped if they have a chance to explore well-chosen contexts thoroughly. Schoenfeld (1994), Kovalainen, Kumpulainen & Vasama (2001) and Streefland & Elbers (1995 & 1997) show another inviting manner. They made the classroom "... a community of mathematical judgment which, to the best of its ability, employs appropriate mathematical standards to judge the claims made before it." (Schoenfeld, 1994, p. 62).

In the case of fractions, for example, Behr et al. (1983), Streefland (1982 & 1990), Watson, Cambell and Collis (1993), Tzur (1999) and Mack (1990 & 2000) show how the construction of fractions by students can originate in the solving of meaningful problems. Watson, Cambell and Collis (1993) show that offering students open problem situations, where no solution is at hand, may result in several approaches on various levels. Watson et al. point out that, in this way, justice is done to the potentials of all students (cf. Freudenthal, 1973).

Although meaningful situations help low-achievers to acquire mathematical knowledge, their high achieving peers do a much better job (cf. Keijzer & Terwel, 2000). The "landscape of learning" metaphor used by Fosnot and Dolk (2001) provides an appropriate way of depicting the way students learn. Fosnot and Dolk consider learning as making a journey through the landscape. Different students take different routes on their way to the horizon, each having their individual experiences. In this metaphor, the low-achieving students are the ones who leave the main route, and get lost in the realization that they are without adequate means to find the way back and that some parts of the terrain will remain closed to them.

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[1] Translated from the original Dutch text.

[2] Translated from the original Dutch text.

Research Questions

By and large, there is a considerable body of evidence to justify the conclusion that a problem-solving approach in teaching can implicate difficulties for low-achievers. This effect is strengthened by the fact that the teaching here concerns fractions, one of the most difficult subjects in primary school (Hasemann, 1981). However, other researchers show that when mathematics education is aimed at learning in meaningful contexts, there is little room for a curriculum design in which mathematics consists in merely following incomprehensible rules (Treffers, 1987; Freudenthal, 1991; Gravemeijer, 1994; Romberg, 1994; Schoenfeld, 1994). The basic issue in the development of mathematics education can therefore be stated as follows: how can we adopt a problem-solving approach to teaching fractions in such a way that both low- and high-achievers benefit from this approach? This developmental question was elaborated elsewhere (Keijzer, 1994; Keijzer & Lek, 1995; Keijzer & Buys, 1996; Keijzer, 1997; Keijzer & Terwel, 2000; Keijzer, 2003) and resulted in the curriculum we described earlier, in which well-chosen contexts are discussed and subsequently elicit the number line as a model for fractions, which establishes a basis for formal manipulations with fractions.

Here we focus on the learning processes of Shirley, a low-achieving student in mathematics, and formulate our main research question starting from the perspective discussed in the previous paragraph. However, we are not merely interested in Shirley's learning. In analyzing her learning, within the theoretical framework presented, we look for indications for possible generalizations of Shirley's learning to the fraction learning of other low-achievers. Thus, although the case-study design enables us to carefully analyze Shirley's learning, its generalizability to other students' learning is limited. Considering this, we formulate our main research question in a general sense and specify this question to Shirley's learning in two additional questions:

Do low-achieving students really benefit from a realistic problem-solving approach in acquiring mathematical insights and proficiency in the domain of fractions, and what are the main obstacles in the formalization process from real-life situations to mathematical number sense?

From this overall question we formulated two questions concerning Shirley's learning:

(I) *What are the characteristics of Shirley's learning process in the acquisition of fractions?*

(II) *What are the key processes showing that Shirley's learning process develops less well or not at all, in particular with regard to fractions?*

We will look at Shirley's choice and use of strategies in problem solving and how she copes as a low-achiever in the mathematics classroom. From other research conducted in this field we learn that low-achievers in mathematics are in an unfavorable position when learning formal fractions. They experience difficulties in problem solving tasks (Verschaffel, 1995), which can conclude in failing to learn fractions and desperately trying to cope with the situation by using counter-productive strategies (Holt, 1964).

METHOD

In this study we closely follow the learning process of one student. Several researchers show the impact of a case-study design as a means to reveal the effects of fraction teaching programs (e.g. Erlwanger, 1973; Carraher & Schliemann, 1991; Mack, 1990; Mack, 1995; Mack, 2000; Hunting, 1983, Tzur, 1999).

In the previous paragraphs we provided several elements of the theoretical framework underlying the present study. This framework includes the theory of realistic mathematics education, RME (Treffers, 1987; Van den Heuvel-Panhuizen, 1996), as it has developed in the Netherlands over the past 30 years. Moreover, it includes the notion of learning mathematics as a social enterprise (Schoenfeld, 1992; Schoenfeld, 1994; Romberg, 1994; Cobb & Whitenack, 1996; Greeno, 1997). And finally it includes theoretical notions on the learning of fractions by the way of modeling well-chosen contexts (Streefland, 1983; Streefland, 1990; Treffers, 1987; Freudenthal, 1991).

We collected our data on the development of Shirley in three different ways. First, we audiotaped and observed all the lesson on fractions Shirley attended during the year the research took place. We elaborated the data for each lesson into a report containing both essential narratives about the things that went on during the lesson, as well as protocols of relevant student-student and student-teacher interactions. The combination of teacher and researcher roles helped us in valuing the qualitative data (cf. Tzur, 1999). In addition, we tested Shirley three times using standardized tests to establish her skill in "numbers and operations" and "measurement and geometry" (Janssen, Kraemer & Noteboom, 1995). Furthermore, in three standardized interviews we determined Shirley's knowledge of fractions. All interviews were audiotaped and transcribed for further elaboration. Elsewhere we provided an extensive description of the interviews and the measures we took to standardize the interviews (Keijzer & Terwel, 2000; Keijzer & Terwel, 2003).

Yin (1984) states that one of the main problems in performing a case study is to organize the large amount of research data that is generated during the inquiry. In this study our data include protocols of all the fraction lessons in grade 6, results of general mathematics tests, and accounts and analyses of standardized interviews. To make it presentable we adapted this material in several steps. First, we rewrote the protocols and our analyses of the interviews as narratives (cf. Gudmundsdottir, 1995) which tell the story of Shirley's progress with fractions. We then ordered these narratives in such a way that they show key moments in Shirley's development and clarified how she constructed her fraction knowledge. After having ordered these key elements we labeled the stages in Shirley's learning of formal fractions as follows: "acquisition of fraction language," "process of formalization," and "dropping out." In addition, we used the labels of the stages to reduce the number of narratives. From the narratives in the first stage of the program we selected those that clearly show the improvements in Shirley's mastery of fraction language. Similarly, from the second stage we selected the narratives that give a clear example of Shirley's struggle with formal fractions; from the third stage we chose the narratives which show how Shirley signals that she is dropping out.

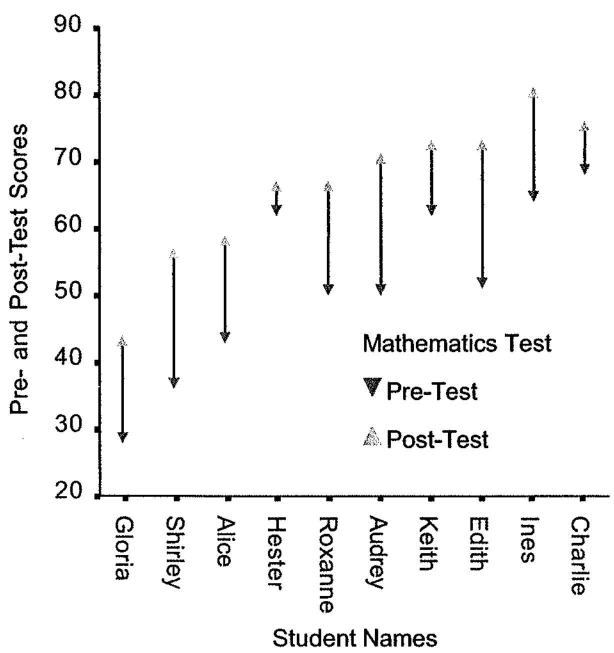
Data and Analysis

Introduction

As stated before, Shirley, being a low-achiever, belongs to the 25 percent weakest in mathematics in her age group as judged within a national context. From the diagram that displays the pre- and post-test scores on the general mathematics test recorded at the start and the end of the case study, we read Shirley's progression in this one-year period (Figure 1).

FIGURE 1

Development of number skills of Shirley and her classmates over the year reported upon in this study.



Scores adapted from Janssen, Kraemer & Noteboom (1995).

Here we present a description of Shirley's learning of fractions in the sixth grade (9-10 year). Since we wanted to disclose the learning process of a low-achiever in the sixth-grade curriculum, Shirley participated in the program for the whole school year. Her test scores, which did not differ that much from the average student, suggested to us that Shirley should have been capable of following at least a major part of the developed program. However, our choice implied that Shirley would remain in the program even if that would not be the case in normal teaching. We did not offer Shirley special treatment or extensive help, since we wanted to find out how she would develop in the fraction program without such treatment or help.[3]

We decided to conduct our research in this way in order to provide arguments concerning the extent to which low-achievers should have to learn formal fractions at all in primary school. In normal school practice there is only a limited amount of time for fraction programs. Bokhove et al. (1996) in their program the Fractiongazette suggest about 80 fraction lessons in grade 6, 7 and 8 (9-12 year). The most recent Dutch textbook series spends about this time on the teaching of fractions (e.g. Huitema (ed.), n.y.). Moreover, this limited attention to fractions is in line with Dutch curriculum standards (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994). And, if this limited time is not sufficient to teach fractions in a meaningful manner, there would seem to be a good reason for reconsidering educational priorities.

Following Yin (1984) we established three recognizable stages in the development in Shirley's learning of fractions. Initially, Shirley is involved in the process of learning the language of fractions. Subsequently, relations between fractions are explored. Here we see, however, that failure leads to the drop-out process.

Acquisition of Fraction Language

From the beginning, the fraction program in which Shirley was involved paid considerable attention to the learning of fraction language. Many researchers underline the importance of the knowledge of fraction language as a basis for forming proper and extended fraction concepts (Bezuk & Bieck, 1993; Connell & Peck, 1993; Mack, 1995; Mack, 2000; Streefland, 1983; Streefland, 1990). Others show how students get stuck during (formal) operations with fractions when their fraction language is not firmly based (Clements, 1980; Hunting, 1983).

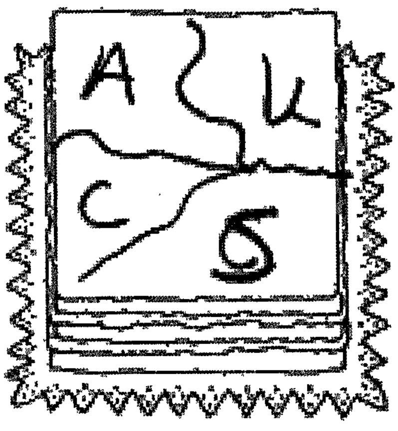
[3] We were well aware that in the research we were responsible for Shirley's development. Of course, we did not want to do anything that might harm her and we would have removed her from the program immediately.

When we observe Shirley's acquisition of fraction language, the first thing we notice is the difficulty she experiences in dividing an object into equal parts, or parts equal in size. In Shirley's first lesson we introduced a context in which cakes have different toppings. One of the students' tasks is to make a cake with four toppings of equal size. Though the equal size of the different parts of the cake is emphasized in discussing the situation with the students, Shirley produces the division depicted in Figure 2. In her attempt to divide into four equal parts, Shirley at first separates topping K, then marks the position of taste A, and finally makes C and S. Unlike many of her classmates, Shirley does not divide by drawing two perpendicular lines. This indicates the first clear distinction between her approach and that of most of her peers. Drawing two lines, as most students did, can be seen as a prelude to understanding the numerical relation between a half and a quarter, 1/2 = 2/4.

When discussing the sharing context in a whole class discussion introducing the student's tasks, Shirley unmistakably brought forward that divisions made should result in equal sized parts. We noticed that here the context largely determined Shirley's approach. When asked to make sure the parts are of equal size, she suggests the perpendicular division as a solution to the problem: "You just need to make a stroke this way (Shirley indicates a horizontal line) and this way (Shirley indicates a vertical line)." However, when working individually, the context probably is not that evidently present. We assume Shirley reads the problem in her task as dividing in any four parts that are about equal in size, leading to the division presented (cf. Weade, 1995).

FIGURE 2

Shirley's cake with four toppings, which should be divided into parts of equal size.



In the following lessons, we see how Shirley meets with considerable difficulties in naming fraction parts. In her fourth lesson she names most parts as "a quarter" or just "piece." To Shirley, the situations presented could be dealt with using just the words "piece" and "quarter" as equivalent terms and did not stimulate her to build up a more developed fraction language. By that time, when most of Shirley's classmates know how to name fractions which move forward as parts of a folded bar, Shirley denominates parts-wholes in terms of halves and quarters only.

Shirley's first ten lessons in fractions were mainly devoted to helping her to develop a fraction language. After these ten lessons she was interviewed. One of the interview problems is about a partly painted wall. In a picture we showed Shirley, 5/6ths of the rectangular object were shaded. We asked Shirley to name the part painted in terms of fractions.

Shirley restricts herself to an informal fraction name and calls the shaded part "three quarters." When we ask her to write this down, she gets a little confused. Here a more advanced fraction languages is needed, but not available. At first she names the fraction "3 quarter," but Shirley is not sure this is the right answer, so she next tries "3/1" and "1/3."

After this interview, the lessons gradually focus less on the acquisition of fraction language. The aim in the next ten lessons is to develop the number line as a model for fractions and to form several strategies in comparing fractions. These comparison strategies for most of Shirley's classmates gradually develop into formal reasoning with fractions. Shirley, however, experiences difficulties in grasping the strategies involved in comparing fractions and more and more uses generalized, but uncomprehended numerical relations. For example, she constructs fractions equivalent to 2/3 by imitating her classmates in doubling of numerator and denominator to thus produce the equivalent fractions 2/3, 4/6 and 8/12. Moreover, she starts guessing answers. After the second series of ten lessons we interviewed Shirley a second time.

In this interview we use a bicycle-tour context to ask Shirley to compare the fractions 2/3 and 5/6. Here we observe how Shirley is unable to position 2/3 on the line depicting the tour. Moreover, her strategy in comparing 2/3 and 5/6 shows that Shirley has a poor understanding of fractions.

Interviewer: "Who do you think got further?"
Shirley: "Janneke."
Interviewer: "Why do you think that?"
Shirley: "Because 5/6 is greater than 2/3."
Interviewer: "Can you explain a little more?"
Shirley: "Well, because that is sixth and that makes larger pieces than thirds."

So, according to Shirley, 5/6 is greater since the pieces in 5/6 are bigger than those in 2/3. Later in this interview we observe two approaches to compare fractions side by side. First, Shirley, the same way she compared 2/3 and 5/6, focuses only (and wrongly) on the denominator of the fractions involved. Secondly, she decides on the size of the fraction by considering both numerator and denominator. The greater these two numbers are, the greater the fraction. She seems to be unable to generalize the equivalence of the fractions 2/3 and 4/6 she found just two weeks earlier. Moreover, when she is searching for fractions close to 3/4, her approach becomes even clearer. According to Shirley 3/3 and 3/5 are equally far from 3/4, as they are both "one away from 3/4," again she refers to pieces, no matter from what division these originate.

The following lessons were aimed at further development of the number line. This served two objectives. By considering fractions at the same position on the number line, equivalent fractions emerged, clearing the way to reaching more formal relations between fractions. On the other hand, positioning fractions on the number line provided the students with an opportunity to reconsider strategies in comparing fractions. Thus, on several occasions after her second interview we discussed comparing strategies with Shirley. However, at the end of one year of fraction learning, after 30 forty-five minute lessons in fractions, Shirley still encounters difficulties in comparing simple fractions. Moreover, she experiences some difficulties in dividing objects when the division is described in terms of fractions and in the (exact) positioning of fractions on a number line. She, however, developed a feeling about where some simple fractions should be located, like she now recognizes that fractions where the numerator is one less than the denominator, like with 5/6, are not that far away from 1.

On the whole, we observe that emphatic attention to the learning of fraction language did not result in Shirley's developing a firm grip on the language of fractions. She used informal fraction-names for an extended period of time and it took her a very long time to use formal fraction-names next to the informal ones. Moreover, Shirley continuously struggled with the meaning of fractions and from time to time treated them as two whole number pairs.

Process of Formalization

In our program of fractions, well-chosen contexts elicit fraction language, followed by operations with fractions. Fractions hereby transform from describers of recognizable situations to formally embedded mathematical objects (Bergeron, Herscovics & Bergeron, 1987; Dubinsky, 1991; Freudenthal, 1973; Hart, 1987; Krutetskii, 1976; Moore, 1994; Piaget, 1973; Streefland, 1987; Streefland, 1997). The learning of fractions may thus be regarded as a process of formalization.

For Shirley, too, interpreting recognizable contexts forms the start of the fraction learning process. In her second lesson we ask Shirley to divide a sausage into four parts. Shirley does so by halving the sausage twice. In her third lesson, the context of baker Bas is reintroduced. This baker prepares fruit tarts with different toppings. Shirley works on the problem of preparing a fruit tart with 1/3 pineapple, 1/3 berries and 1/3 kiwi. She divides the tart into nine pieces and makes three pieces of kiwi (K), three pieces of berries (B) and three pieces of pineapple (A).

Shirley interpreted the situation as such, that dividing in many parts could facilitate making divisions. And, in some sense these constructions in Shirley's second and third lessons look promising, in view of the intended process of formalization.[4] Let us therefore turn to more formal situations to see how Shirley uses and generalizes these potential equivalence relations.

As we saw, after about twenty lessons Shirley did not use the equivalence of 2/3 and 4/6 to compare 2/3 and 5/6. In her second interview we asked Shirley to arrange the fractions in three groups: smaller than one half, exactly one half and greater than one half. Unlike her classmates, Shirley did not halve the denominator to compare this result with the numerator, although this strategy could be expected from Shirley's approach when dividing a sausage or fruit tart. The context of the fruit tart probably helped her to understand the fractions involved, and supported her in visualizing the fractions, which in turn created an overview of the situation. However, in comparing bare fractions, she seems to place fractions with large denominators and numerators in the group exceeding one half. More generally, we observed that Shirley compared fractions by looking at the size of the denominator and the numerator. However, when she was asked to use a bar while comparing the fractions, she replaced this approach by reading the conclusion from the two divided bars.

We further observed that Shirley had difficulties with fractions in situations that are less familiar to her. In the fifteenth lesson we introduced the context of the fraction-lift.[5] Here the vertical number line represents a so-called "fraction house," which houses a number of fractions. Lifts connect the different floors in the building. The numbers of the lifts indicate the stops they make: for instance, the 3-lift stops three times, at 1/3, 2/3 and at the top of the building (at 1). Similarly, the 4-lift stops at 1/4, 2/4, 3/4 and at 1, the 2-lift stops at 1/2 and at 1, et cetera (Figure 3). This context thus makes explicit the different fractions belonging to the same position on the number line.

The fraction-lift was developed as context, where numerator and denominator could be considered separately and where fraction positioning on a number line – by means of the lifts – made considering equivalent fractions necessary. Moreover, the fraction-lift provides for a language to consider fraction positions and therefore facilitates fraction operations (cf. Sfard, 1994). This provides opportunities for students to explore the situation by taking the role of the fractions involved (cf. Tuyay, Floriani, Yeager, Dixon & Green, 1995). However, the result of this context-construction is also rather abstract in the sense that it is hardly embedded in meaningful experiences apart from those within mathematics as abstract structure itself. We saw that especially low-achievers could be disadvantaged in this situation.

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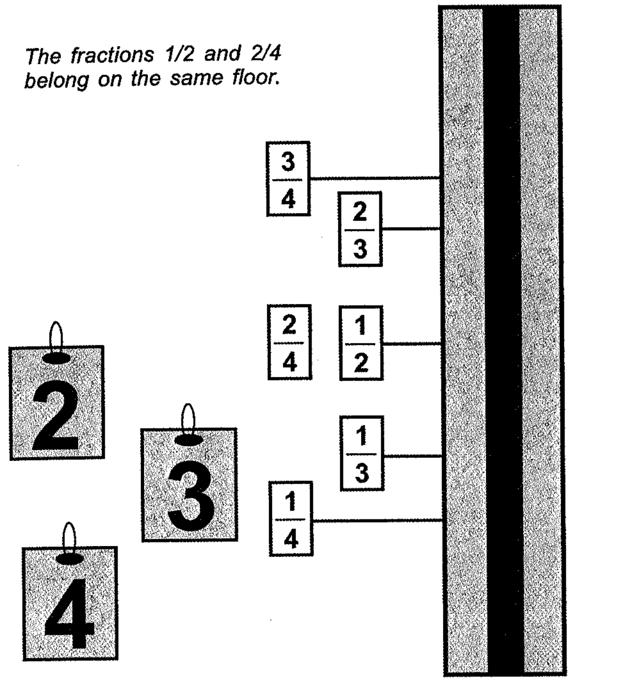
[4] However, the division of the fruit tart could well be the result of a misconception of the problem. Shirley here could have interpreted that 1/3 pineapple, 1/3 berries and 1/3 kiwi meant 3 pieces of each, since three pieces is her meaning of 1/3.

[5] The fraction-lift is an idea of Adrian Treffers.

FIGURE 3

The fraction-lift, with 2-lift, 3-lift and 4-lift.



In the fifteenth lesson we discuss fractions which live at the top of the building. All students know that 4/4 lives at that highest position. Shirley is eager to name another fraction at the top: 8/8. She shows she sees regularities and tells the 8-lift will go to ½ and so does the 16-lift. Later in a similar situation we ask Shirley where the 3-lift stops.

Teacher: "Shirley, where do you think this lift stops first."
Shirley: "At one quarter."
Other students (whispering): "One third."
Shirley (aloud): "One third."
Teacher: "Let's take the 9-lift now. Do you think this one will stop at one third too?"
Shirley: "Yes...eh...no...eh...no, it does not."

What we see here typifies Shirley's way of dealing with fractions in non-familiar or artificial contexts. If she is asked to show the meaning of fractions within a flexible context, she fails. However, if she recognizes regularities in the numbers, she is eager to bring in several other examples. Here also, the strategy, which in general can be described as "doubling," is Shirley's favorite, where she once again focuses on the uncomprehended numerical relations. Moreover, like many low-achievers, she holds on to these results without questioning these or searching for more meaningful extensions.

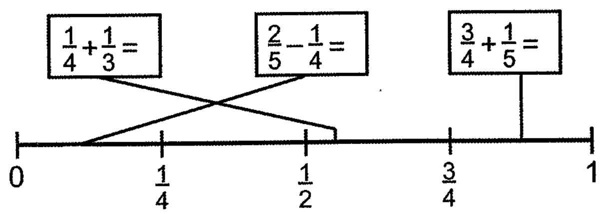
In the seventeenth lesson Shirley shows that a few of the fraction relations she constructed by doubling both numerator and denominator became "fraction-facts," to be used in suitable situations. In this lesson we introduce a computer version of the fraction-lift. While playing the game, Shirley shows she knows that 5/10 is halfway up the lift-line and that placing a fraction there can be helpful. Another observation, however, shows the important disadvantage of relying solely on memorized fraction-facts. In the fourteenth lesson we discuss a school journey with the students. In this context the children are allowed to choose between two attractions. We tell the students that 3/5 of the students chose the first attraction and 2/5 the second. Shirley explains why she thinks more children chose the first attraction. She points at the fraction 2/5: "I know this is a half and the other one is more than a half." Shirley memorized the equivalence of 2/5 and a half and then used this "fact" in her reasoning.

In the discussion following Shirley's explanation, Shirley's classmates try to convince her that 2/5 does not equal 1/2. Charley explains: "2/5 is less than a half. 2½/5 is a half." He draws a bar on the blackboard to clarify his conclusion. However, although we saw frequent examples where Shirley supported her own reasoning by using a bar, it now seems very difficult for her to use Charley's reasoning in reviewing her answer. Moreover, Shirley has not developed a mechanism to check her answer by using her knowledge of fraction language (2/5 meaning 2 of 5 pieces), nor did the situation invite her to make her own drawing. In addition, she does not seem to be willing to review her answer once she has found a way to solve a problem.

After about twenty-five lessons, Shirley considers fractions as formal objects where the denominator decides the kind of object. This supports her in adding and subtracting fractions with equal denominator. She uses this knowledge, together with known relations between simple fractions like ¼, ½ and ¾, to roughly position the result of sums like 2/5 – ¼ and ¾ + 1/5 on a number line (Figure 4). In doing this, Shirley, depending on her strategy, shows a reasonable knowledge of the size of the fractions involved.

FIGURE 4

Placing 2/5 – 1/4 and 3/4 + 1/5 on a number line.



On the whole we see that Shirley has major difficulties in explaining her approaches and inclines towards instrumental understanding (cf. Booth & Thomas, 2000). In familiar contexts, presented visually, we saw that Shirley was able to interpret the situation and could solve connected problems. But we also found how Shirley experienced difficulties in interpreting the visually presented fraction-lift. Our observations and those of other researchers suggest that familiarity with the context is a more important key to success than the manner in which the problems are represented (Kraemer, 2000; Featherstone, 2000; Greer & Harel, 1998; Koch & Li, 1996; Mack, 1990; Streefland, 1982).

Facing problems

In the thirteenth lesson we present the context of a painting contest. We tell the students that 600 children participated in the contest. We use a bar to depict the 600 children, when we discuss what number of children used a felt pen (1/2 of the participants), what number of children were 4 and 5 years old (1/5 of the participants in the contest), et cetera.

At the end of the lesson we explore divisions in a bar while dealing with fractions with denominator 10. The results are represented on a double-indexed bar. On the bar we write the number of children that fits with 1/10 and 2/10. When it is Shirley's turn, she is asked what number of children is 3/10 of the group of 600.

Shirley knows she has to add 120 and 60 to solve this problem, but experiences great difficulties in doing so.

Shirley: "Ah...240."

Teacher: "Please try one more time."

Shirley: "..."

Teacher (points at the drawn bar): "1/10 of the children equals 60 children, 2/10 equals 120, 3/10 equals ..."

Shirley: "It's 240!"

Other student (whispering): "180."

Shirley (repeats): "180."

From this moment onwards, Shirley experiences difficulties in working with fractions as a result of her weakness in basic number strategies and especially as a result of her incapacity to see multiplicative number relations. Moreover, from this moment too, there is a shift from developing models from situations to using models for situations where formal manipulations with fractions are needed (Gravemeijer, 1994). These two elements of the fraction program in which Shirley is involved signify the start of facing major problems in learning fractions. In the next lessons Shirley signals her dislike of the fraction program. Frequently she scamps her work and on some occasions, when she is working individually, she does not want to be helped by her fellow students or the teacher. She copies her answers from her neighbors and yells out numbers at random to answer questions during the lessons.

Our observations of Shirley are consistent in this respect with findings by Deal et al. (2000). In their case study they describe the development in reasoning of Reed, a low-achiever in mathematics. Deal et al. found that Reed was unable to construct reasoning on more formal levels. Moreover they state: "Reed remained hesitant throughout the study, despite the play-like atmosphere and the research team's frequent visits to the school," (Deal et al., 2000, p. 25). Under similar conditions we observed the same hesitant reactions from Shirley.

Summary

The fraction program followed by Shirley started with situational contexts which evoked fractions. Shirley seemed able to deal with the problems as long as informal answers were a possibility. At the same time, she started to construct formal fraction language when negotiations in the classroom forced her to do so. While doing so, Shirley experienced her first difficulties in the acquisition of fraction language. It took her quite some effort to get the idea of pieces of equal size – as fraction characteristic and not one imposed by the individual context only. However, her proficiency gradually developed and after 30 lessons was able to (instrumentally) add fractions with identical denominators, while still encountering problems in translating fraction symbols into divisions on bars, circles or a number line. More generally, Shirley experienced obstacles in situations where she was asked to order (mental, schematic or physical) objects.

At the next stage of the curriculum, with situational contexts aimed at positioning fractions on a number line and on comparing fractions, Shirley developed strategies, like doubling and memorizing fraction facts, that can be characterized as instrumental understanding. In manipulating fractions, she generalized number patterns without realizing how these referred to the situational contexts underlying the number patterns. This answers our research question (I) concerning the characteristics of Shirley's formal fraction learning process.

When unfamiliar situations are applied, in which real understanding of the formal nature of fractions and fraction language is needed, Shirley starts dropping out. The process is strengthened by Shirley's limited number strategies and also by her lack of enthusiasm for the topic of fractions. Shirley gradually developed several coping strategies in handling fractions. These strategies constitute an answer to research question (II) concerning key processes showing how Shirley's learning process develops less well, or not at all. These strategies include:

- use drawings and informal approaches to deal with fractions,
- generalize simple numerical relations that come forward,
- when no way can be found to reflect on her answer, state that the given answer is correct,
- when mathematical connections cannot be made, yell out answers,
- when problems do not make sense, copy answers from others or use the back of the worksheet to make drawings, especially when the context encourages you to do so (cf. Schoenfeld, 1992, p. 359).

During the research year we found that Shirley gradually and consequently grew in her knowledge of fractions and observed her – in some sense adequate – coping strategies. This convinced us that we could go along with Shirley, to see how she liked to draw on the back of her worksheet, how she made others laugh when yelling an answer and how her friends liked to help her with their answers to protect her from failure.

Shirley – in some sense – failed to cope with our fraction program, but was able to survive in her group (cf. Holt, 1964). We therefore found a more or less negative answer to our main research question. We were not able to adapt a problem-solving approach in teaching fractions in such a way that both low- and high-achievers could benefit from the approach.

DISCUSSION AND CONCLUSION

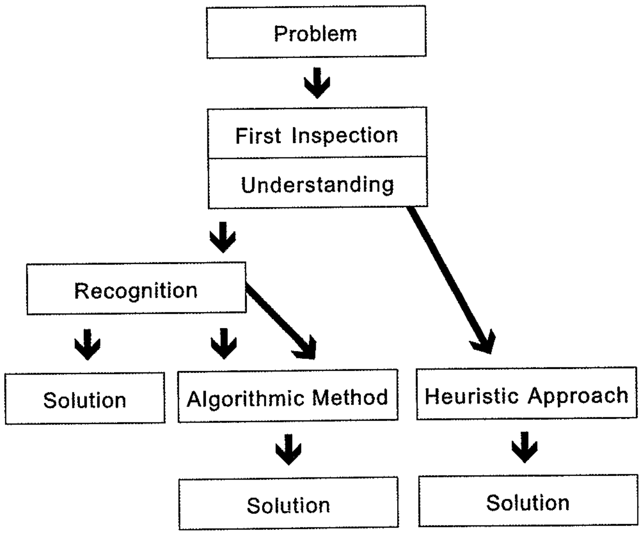
Our point of departure for the study presented was the following question: *Do low-achieving students really benefit from a realistic problem-solving approach in acquiring mathematical insights and proficiency in the domain of fractions, and what are the main obstacles in the formalization process from real life situations to mathematical number sense?*

In the case study described here, we found that Shirley, a low-achiever in mathematics, experienced several difficulties in learning fractions as part of our program. We analyzed the obstacles Shirley found on her way and established from the extensive literature on learning mathematics that Shirley's development is typical for low-achievers. In this way we found indications that low-achieving students use at least two kinds of strategies: constructive and disruptive strategies, that is productive and counterproductive strategies. The constructive strategies are the ones mentioned by Alexander, Graham and Harris (1998) and Siegler (1991) in their analysis of cognitive strategies. In this category we encounter, for example, the strategy of drawing the situation and using a model, drawing a bar or dividing a circle. However, Shirley, like many low-achievers, often uses disruptive, counter-productive, self-defeating and (even) self-handicapping strategies. These strategies include taking a wild guess and seeing what happens, making the teacher do the job, cheating, and copying answers without understanding. The students thus express anxiety and a fear of failing, which, in turn, is a threat to their self-esteem. In the course of Shirley's year-long learning process we see a kind of shift from productive strategies towards counterproductive ones, and even disruptive strategies as described so vividly by Holt (1964). There is a limit to the effectiveness of providing low-achieving students with strategies and models for handling problems with fractions. The transmission of techniques for thinking and problem solving falls on barren ground unless anxiety can be reduced and children are given more time to explore fractions in familiar contexts, in a more relaxed pace, under the guidance of the teacher and in interaction with more able peers (Schoenfeld, 1992; Greeno & Goldman, 1998). We argued that the limits inherent in teaching primary school mathematics to low-achievers provide a convincing argument for setting different priorities in the teaching context. Furthermore, spending more time on the teaching of fractions, for example by devoting special attention to low-achievers, in this case, appears not to be an appropriate choice. Since we are dealing with low-achievers in mathematics, there are probably other topics requiring more serious consideration with regard to how educational time should be spent.

In analyzing Shirley's learning process we observed her fraction-learning process in depth. We saw how difficulties arose due to Shirley's limited knowledge of number relations, her uncertainty in representing problems and her lack of reflection on her work. In this connection, let us look at Van Streun's (1989) schematized problem solving. In his schematic representation (Figure 5) we can track Shirley's approach in solving problems with fractions. When her first inspection leads to the conclusion that she does not recognize the problem, she drops out. However, when the first inspection leads to the recognition of a known problem – that is, in Shirley's perception – she then assumes she knows the answer at once, or starts some (usually erratic) algorithmic approach. In other words, Shirley ends up on the left side of Van Streun's diagram, while teaching was concentrated on the right side, that is, on heuristic approaches. This adds another answer to the question of why low-achievers experience so much difficulty in the program described.

FIGURE 5

Scheme for solving problems, adapted from Van Streun (1989, p. 17).



Sweller (1994) adds yet another explanation for why fractions should be so troublesome for Shirley and low-achievers like her. Sweller describes how a program such as the one developed could easily cause "cognitive overload" in low-achievers. Sweller proposes to reduce cognitive load by means of improved isolating skills and strategies. However, as Schoenfeld (1994) points out, the interconnectivity is inherent in learning mathematics since it is about mathematization, abstraction and understanding structure:

> Learning to think mathematically means (a) developing a mathematical point of view – valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure – mathematical sense-making. (Schoenfeld, 1994, p. 60)

We only analyzed the learning of one student. We argued therefore that our conclusions cannot be easily generalized to the whole group of low-achievers. However, as the observed patterns in Shirley's learning are typical for many low-achievers, we are convinced that Shirley and low-achievers like her are trapped, on the one hand, by the nature of the mathematics learning as proposed by Schoenfeld and as elaborated by us in our fraction program, and, on the other hand, by the limits of their mathematical abilities.

If we want to establish "mathematics for all" we should set priorities for all students. In the case of low-achievers this might well lead to limited attention to fractions in order to enable these students to develop the mathematics that suits their aptitudes (cf. Kraemer, Van der Schoot & Engelen, 2000). Or, as Doornbos (1997) pleads:

> In primary education...an exhaustive list of unequivocally formulated standards – aims for all students to be pursued – is superfluous and mistaken. We are talking about the education of children of school age. Also, children who experience temporary learning difficulties, or whose ability to learn is limited should be made to feel welcome, without being discriminated against. (Doornbos, 1997, p. 26) [6]

Shirley and other low-achievers should feel accepted at their school. Teaching formal fractions and requiring her to discuss formal relations that are obscure to her, and forcing her to construct models that do not help her to gain the required insights should not be part of her curriculum.

We propose that uniform standards be reconsidered and that we abandon the idea in primary education that, with some exceptions, all students be required to learn the same things. Students who cannot learn formal mathematics should be welcomed and experience mathematics they can understand and use in daily life. The policies advocated by those in the public arena who talk of uniform standards should be regarded as unrealistic and even counterproductive.

We need in mathematics a "sounder" model of learner growth and academic development (Alexander, 2000). This point of view is not only based on the experience of the large differences in the acquisition of mathematics observed in various studies in mathematics education (Terwel, 1990; Hoek, Terwel & Van den Eeden, 1997; Hoek, Van den Eeden & Terwel, 1999), but also on the theories and views of scholars who have encountered individual students like Shirley in their research, in their classes or in their tutorial interactions (Davis, 1994; Freudenthal, 1973, 1991; Doornbos, 1997; Gravemeijer & Terwel, 2000).

Our views should not be understood as a plea for early selection, ability grouping or streaming. On the contrary, in our opinion the issue of how to organize teaching in such a way that all students can benefit is still very much open to resolution (see Keijzer, 2003). Finally, we agree with Freudenthal (1973), who was strongly committed to mathematics as a human activity, that both high and low-achieving students should be included in the community of learners.

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[6] Translated from the original Dutch text.

REFERENCES

Alexander, P. A. (2000). Toward a Model of Academic Development: Schooling and the Acquisition of Knowledge. *Educational Researcher, 29*(2). 28-34.

Alexander, P.A., Graham, S. & Harris, K.R. (1998). A Perspective on Strategy Research: Progress and Prospects. *Educational Psychology Review, 10* (2). 129-151.

Behr, M.J., Lesh, R., Post, T.R. & Silver, E.A. (1983). Rational-Number Concepts. In Richard Lesh and Marsha Landau (Eds.). *Acquisition of Mathematical Concepts and Processes*. New York/London: Academic Press. 91-126.

Bergeron, A., Herscovics, N. & Bergeron, J.C. (1987). Kindergartners' knowledge of numbers: A longitudinal study. Part II: Abstraction and formalization. In J.C. Bergeron, N. Herscovics, C. Kieran (Eds). *Proceedings of the eleventh international conference Psychology of Mathematics Education*, Vol. II. Montreal. 352-360.

Bezuk, N.S. & Bieck, M. (1993). Current Research on Rational Numbers and Common Fractions: Summary and Implications for Teachers. In Douglas T. Owens (Ed.), *Research Ideas for the Classroom - Middle Grades Mathematics.* New York: McMillan. 118-136.

Bokhove, J., Buys, K. (Ed.), Keijzer, R., Lek, A., Noteboom, A. & Treffers, A. (1996). *De Breukenbode. Een leergang voor de basisschool* (werkbladen en handleiding) [The Fractiongazette]. Enschede/Utrecht: SLO/FI/Cito.

Booth, R.D.L. & Thomas, M.O.J. (2000). Visualization in Mathematics Learning: Arithmetic Problem-solving and Student Difficulties. *Journal of Mathematical Behavior 18*(2). 169-190.

Brenner, M.E., Mayer, R.E., Moseley, B., Brar, T., Durán, R., Smith Reed, B., & Webb, D. (1997). Learning by Understanding: The Role of Multiple Representations in Learning Algebra. *American Educational Research Journal 34*(4). 663-689.

Carraher, D. W. & Dias Schliemann, A. (1991). Children's understanding of fractions as expressions of relative magnitude. In Fulvia Furinghetti (Ed.). *Procedings of the fifteenth PME conference (Assisi, Italy)*, Vol. 1. 184-191.

Clements, M.A. (1980). Analyzing children's errors on written mathematical tasks. *Educational Studies in Mathematics 11*. 1-21.

Cobb, P. & Whitenack, J.W. (1996). A Method for Conducting Longitudinal Analyses of Classroom Videorecordings and Transcripts. *Educational Studies in Mathematics 30*. 213-228

Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education] (1994). *Doelbewust leren; kerndoelen in maatschappelijk perspectief* [Purposeful learning; curriculum standards in social perspective]. Den Haag: SDU.

Connell, M.L. & Peck, D.M. (1993). Report of a Conceptual Intervention in Elementary Mathematics. *Journal of Mathematical Behavior 12*. 329-350.

Davis, R.B. (1994). Comments on Thomas Romberg's Chapter. In: Alan H. Schoenfeld (Ed.). *Mathematical thinking and problem solving*. Hilldale, NJ: Lawrence Erlbaum Associates, Publishers. 312-321

Deal, D., Buehl, M.M., Tron, M.G.O., Harris, L., Clements, S., & Hardy, S. (2000). Portraying Mathematical and Analogical Reasoning in the Young: Three Case Studies. Paper presented at the Annual meeting of the American Educational Research Association, New Orleans, April 2000.

Doornbos, K. (1997). *De pedagogische balans* [The pedagogical balance]. Groningen: Wolters-Noordhoff.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In David Tall (Ed.). *Advanced mathematical thinking*. Dordrecht/Boston/London: Kluwer Academic Publishers. 95-123.

Erlwanger, S.H. (1973). Benny's conception of Rules and Answers in IPI Mathematics. *Childern's Mathematical Behavior 1*(2). 7-26

Featherstone, H. (2000) "-Pat + Pat = 0": Intellectual Play in Elementary Mathematics. *For the Learning of Mathematics 20*(2). 14-23.

Freudenthal, H. (1973) *Mathematics as an Educational Task*, Dordrecht, The Netherlands: Reidel.

Freudenthal, H. (1991) *Revisiting Mathematics Education, China lectures*, Dordrecht: Kluwer Academic Publishers

Fosnot, C.T. & Dolk, M. (2001) *Young Mathematicians at Work: Constructing the Number System, Addition and Subtraction*, Portsmouth, NH: Heinemann Press.

Gravemeijer, K.P.E. & Terwel, J. (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies, 32*(6). 777-796.

Gravemeijer, K.P.E. (1994) *Developing realistic mathematics education*, Utrecht: CDß-press

Greeno, J.G. (1991) Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education 22*(3). 170-218.

Greeno, J.G. (1997). On Claims That Answer the Wrong Questions. *Educationals Researcher 26*(1). 5-17.

Greeno, J. G. & Goldman, S.V. (1998). *Thinking Practices in Mathematics and Science Learning*. Mahwah, New Jersey/London: Erlbaum.

Greer, B. & Harel G. (1998). The Role of Isomorphisms in Mathematical Cognition. *Journal of Mathematical Behavior 17*(1). 5-24.

Gudmundsdottir, S. (1995). The narrative nature of pedagogical content knowledge. In H. McEwan & K. Egan (Eds.). *Narrative in teaching, learning and, research*. New York: Teachers College, Columbia University. 24-38.

Hart, K.M. (1987). Practical work and formalisation, too great a gap. In J.C. Bergeron, N. Herscovics, C. Kieran (Eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Vol II*. Montreal. 408-415.

Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics 12*. 71-87.

Hoek, D., Terwel, J. & van den Eeden, P. (1997). Effects of Training in the Use of Social and Cognitive Strategies: An Intervention Study in Secondary Mathematics in Co-operative Groups. *Educational Research and Evaluation 3*(4). 364-389.

Hoek, D., Van den Eeden, P., & Terwel, J. (1999). The effects of integrated social and cognitive strategy instruction on the mathematics achievement in secondary education. *Learning and Instruction 9*(5). 427-448.

Holt, J. (1964). *How Children Fail*. New York: Dell Publishing Co.

Huitema, S., Van der Klis, A., Van de Molengraaf, F., Timmermans, M., & Erich, L. (n.y.). *De wereld in getallen 6. Handleiding a & b* [The world in numbers 6. Teacherguide a & b]. Den Bosch: Malmberg.

Hunting, R.P. (1983). Alan: A case study of knowledge of units and performance with fractions. *Journal for Research in Mathematics Education 14*(3). 182-197.

Janssen, J., Van der Schoot, F., Hemker, B., & Verhelst, N. (1999). *Balans van het reken-wiskundeonderwijs in de basisschool 3* [Balance-sheet of mathematics education in primary school 3]. Arnhem: Cito.

Janssen, J., Kraemer, J.-M. & Noteboom, A. (1995). *Leerling Volg Systeem. Rekenen-Wiskunde 2* [Student Registration System. Mathematics 2]. Arnhem: Cito.

Keijzer, R. (1999). Een klein verschil? [A small difference?]. In W. Faes & W. Oonk (Eds.). *Van Rekenend Nederland voor Fred Goffree. 65/2 Onderwijsverhalen voor pabostudenten.* Groningen: Wolters-Noordhoff. 37-40.

Keijzer, R. (2003). *Teaching formal mathematics in primary education. Fraction learning as mathematising process.* Utrecht: CDß-Press, Utrecht University

Keijzer, R. & Lek, A. (1995). Breuken: kerndoelen en formaliseren [Fractions: curriculum standards and formalising]. In C. van den Boer and M. Dolk (Eds.). *Rekenen in de bovenbouw van de basisschool.* Utrecht: Panama/HvU, Freudenthal Institute. 53-64.

Keijzer, R. & Terwel, J. (2000). Learning for mathematical insight: a longitudinal comparison of two Dutch curricula on fractions. Paper presented at the Annual meeting of the American Educational Research Association, New Orleans, April 2000.

Keijzer, R. & Terwel, J. (2001). Audrey's acquisition of fractions: a case study into the learning of formal mathematics. *Educational Studies in Mathematics 47*(1). 53-73.

Keijzer, R. & Terwel, J. (2003). Learning for mathematical insight: a longitudinal comparative study on modelling. *Learning & Instruction 13*. 285-304.

Keijzer, R. & Buys, K. (1996). Groter of kleiner. Een doorkijkje door een nieuwe leergang breuken [Bigger or smaller. A close look at a new curriculum on fractions]. *Willem Bartjens 15*(3). 10-17.

Keijzer, R. (1994). De ontwikkeling van een breukenleergang - een voorbeeld van ontwikkelingsonderzoek [The development of a curriculum on fractions - an example of developmental research]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs 13*(1). 24-31.

Keijzer, R. (1997). Formeel rekenen met breuken [Formal arithmetic with fractions]. In C. van den Boer and M. Dolk (Eds.), *Naar een balans in de reken-wiskundeles - interactie, oefenen, uitleggen en zelfstandig werken -.* Utrecht: Panama/Freudenthal Instituut. 101-116.

Kerckhoff, A.C. & Glennie, E. (1999). The Matthew effect in American education. In Aaron M. Pallas (Ed.), *Research in sociology of education and socialization (Vol. 12).* Stamford, Connecticut: Jai Press Inc. 35-66.

Koch, L.C. & Li, X. (1996). Students' Understanding of Computation-Related Rational Number Skills. *Journal of Mathematical Behavior 15*. 193-205.

Kovalainen, M., Kumpulainen, K., & Vasama, S. (2001). Orchestrating Classroom Interaction in a Community of Inquiry: Modes of Teacher Participation. *Journal of Classroom Interaction 36*(2). 17-28.

Kraemer, J.M. & Janssen, J. (2000). Druppels op een gloeiende plaat – aandachtspunten voor een grondige discussie over de continue ontwikkeling van rekenzwakke leerlingen tussen vier en zestien jaar [Drops in the ocean – points of attention for a thorough discussion on the continuous development of low-attainers in arithmetic between four and sixteen years of age]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs 18*(3). 3-14.

Kraemer, J.M. (2000). Met handen en voeten rekenen. Naar een natuurlijke benadering van rekenzwakke kinderen [Arithmetic with hands and feet. Towards a natural approach for low arithmetic achievers]. *Vernieuwing. Tijdschrift voor onderwijs en opvoeding 59*(1/2). 29-32.

Kraemer, J.M., Van der Schoot, F. & Engelen, R. (2000). *Balans van het reken- wiskundeonderwijs op LOM- en MLK-scholen 2: uitkomsten van de tweede peiling in 1997* [Balance-sheet of mathematics education in special schools 2: results of the second gauging in 1997]. Arnhem: Cito.

Krutetskii, V.A. (1976 (English edition)). *The psychology of Mathematical Abilities in Schoolchildren* (translated by Joan Teller, edited by Jeremy Kilpatrick and Izaak Wirszup). Chicago/London: The University of Chicago Press.

Lemoyne, G. & Tremblay, C. (1986). Addition and multiplication: problem-solving and interpretation of relevant data. *Educational Studies in Mathematics 17*. 97-123.

Mack, N.K. (1990). Learning fractions with understanding: building on informal knowledge. *Journal for Research in Mathematics Education 21*(1). 16-32.

Mack, N.K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education 26*(5). 422-441.

Mack, N.K. (2000). Long-term effects of building on informal knowledge in a complex content domain: the case of multiplication of fractions. Paper presented at the Annual meeting of the American Educational Research Association, New Orleans, April 2000.

Mcintosh, A., Reys, B.J. & Reys, R.E. (1992). A Proposed Framework for Examining Basic Number Sense. *For the Learning of Mathematics 12*(3). 2-8.

Milo, B.F. (2003). *Mathematics Instruction for Special-Needs Students. Effectes of instructional variants in addition and subtraction up to 100.* Leiden University.

Moss, J. & Case, R. (1999). Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum. *Journal for Research in Mathematics Education 30*(2). 122-147.

Nelissen, J.M.C. (1998a). Representaties in het reken-wiskundeonderwijs [Representations in mathematics education]. *Pedagogische Studiën 75*. 169-183.

Nelissen, J.M.C. (1998b). Taal en betekenis in het realistisch reken-wiskundeonderwijs [Language and meaning in realistic mathematics education]. In N. Boswinkel & M. Dolk (Eds.). *Over rekenen gesproken – taal in/en rekenen.* Utrecht: Panama/Freudenthal Institute. 11-34.

Piaget, J. (1973). Comments on mathematical education. In A.G. Howson (Ed.). *Developments in Mathematical Education, Proceedings of the Second International Congress on Mathematical Education.* Cambridge: Cambridge University Press. 79-87

Pollock, E., P. Chandler & J. Sweller (2002). Assimilating complex information. *Learning and Instruction 12.* 61-86

Reigeluth, Ch.M. (1999). What Is Instructional-Design Theory and How Is It Changing. In Charles M. Reigeluth. *Instructional-design theories and models Volume II. A new paradigm of instructional theory.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers. 5-30.

Romberg, T.A. (1994). Classroom Instruction That Fosters Mathematical Thinking and Problem Solving: Connections Between Theory and Practice. In Alan H. Schoenfeld (Ed.). *Mathematical thinking and problem solving.* Hilldale, NJ: Lawrence Erlbaum Associates, Publishers. 287-304.

Schoenfeld, A.H. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In Douglas A. Grouws (Ed.). *Handbook of research on mathematics teaching and learning.* New York: Macmilian Publishing Company. 334-370.

Schoenfeld, A.H. (1994). Reflections on Doing and Teaching Mathematics. In Alan H. Schoenfeld. *Mathematical thinking and problem solving.* Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers. 53-70.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics 14*(1). 44-56.

Siegler, R. S. (1991). Strategy choice and strategy discovery. *Learning and Instruction 1*(1). 89-102.

Streefland, L. & Elbers E. (1995). Interactief realistisch reken-wiskundeonderwijs werkt [Interactive realistic mathematics education works]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs 14*(1). 12-21.

Streefland, L. & Elbers E. (1997). De klas als onderzoeksgemeenschap [Class as research community]. In C. van den Boer & M. Dolk (Eds.). *Naar een balans in de reken-wiskundeles – interactie, oefenen, uitleggen en zelfstandig werken -.* Utrecht: Freudenthal Institute. 11-24

Streefland, L. (1982). Subtracting fractions with different denominators. *Educational Studies in Mathematics 13.* 233-255.

Streefland, L. (1983). *Aanzet tot een nieuwe breukendidactiek volgens wiskobas* [Impuls for a new pedagogy on fractions according to wiskobas]. Utrecht: OW&OC.

Streefland, L. (1987). Free production of fraction monographs. In J.C. Bergeron, N. Herscovics & C. Kieran (Eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Vol I.* Montreal. 405-410.

Streefland, L. (1990). *Fractions in Realistic Mathematics Education, a Paradigm of Developmental Research.* Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1997). Een geval van reflectief denken in ontwikkeling met verhoudingen als paradigma (slot) [A case of developing reflective thinking with ratio as paradigm]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs 15*(3). 22-31.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction 4.* 295-312.

Sweller, J. (1999). *Instructional designs in technical areas.* Melbourne: ACER.

Terwel, J. (1990). Real Maths in Cooperative Groups in Secondary Education. In N. Davidson (Ed). *Cooperative Learning in Mathematics.* Menlo Park: Addison-Wesley. 228-264.

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project.* Dordrecht: Reidel Publishing Company.

Tuyay, S., Floriani, A., Yeager, B., Dixon, C., & Green, J. (1995). Constructing an integrated, inquiry-oriented approach in classrooms: A cross-case analysis of social, literate, and academic practice. *Journal of Classroom Interaction, 30*(2). 1-15.

Tzur, R. (1999). An Integrated Study of Children's Construction of Improper Fractions and the Teacher's Role in Promoting That Learning. *Journal for Research in Mathematics Education 30*(4). 390-416.

Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education.* Utrecht: CDß-press.

Van Lieshout, E.C.D.M. (1997). What can research on word and context problems tell about effective strategies to solve subtraction problems? In M. Beishuizen, K.P.E. Gravemeijer and E.C.D.M van Lieshout. *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures.* Utrecht: CDß-Press. 79-111.

Van Luit, J.E.H. & Van Rijt, B.A.M. (1997). Stimulation of early mathematical competence. In M. Beishuizen, K.P.E. Gravemeijer & E.C.D.M. van Lieshout (Eds.). *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures.* Utrecht: CDß-Press. 215-237.

Van Streun, A. (1989). *Heuristisch wiskunde-onderwijs – verslag van een onderwijsexperiment* [Heuristic Mathematics Education]. Groningen: Rijksuniversiteit Groningen.

Verschaffel, L. (1995). Beïnvloeden van leerprocessen [Influencing of learning processes]. In: J. Lowyck & N. Verloop (eds.). *Onderwijskunde. Een kennisbasis voor professionals.* Groningen: Wolters-Noordhoff. 153-187.